3. CAPITAL ACCUMULATION AND GROWTH: THE BASIC SOLOW MODEL

■ The previous lecture raised a basic economic question: **how can a nation escape from poverty** and ultimately become rich? Or more precisely: **how can a country initiate a growth process** that will lead it to a higher level of GDP and consumption per person?

■ This lecture presents a **fundamental economic model that delivers some first answers**. The model will show **how the long-run evolutions of income and consumption per worker** in a country are **affected by structural parameters** such as the country's rate of **saving** and investment and the growth rate of its **population**. The model is known as the **Solow model of economic growth**.

■ In 1956 the economist and Nobel Prize Laureate Robert M. Solow published a seminal article "A Contribution to the Theory of Economic Growth", *Quarterly Journal of Economics*, 70, which presented a coherent **dynamic model with an explicit description of the process of capital accumulation** by which saving and investment become new capital.

■ In the Solow model, **competitive clearing of factor markets** implies that **output** in each period is **determined by the available supplies of capital and labour**. Furthermore, total **saving and investment** is assumed to be an **exogenous fraction of total income**, and the **labour force** is assumed to **grow at a given rate**.

■ The essential additional **feature of the Solow model** is that it incorporates the **dynamic link between the flows of savings and investment and the stock of capital**. Solow's model accounts for the fact that between any two successive periods, the **stock of capital will increase by an amount equal to gross investment minus depreciation** on the initial capital stock.

■ The model describes **how capital evolves as a result of capital accumulation**, **how the labour force evolves as a result of population growth**, and **how total production and income evolves as a consequence of the evolutions of the total inputs of capital and labour**.

■ The model therefore involves a certain **evolution of income per worker as well**. It thereby contributes to answering the fundamental question of what determines "the wealth of nations".

■ This lecture presents the **Solow model in its most basic version**. Considering the simple framework, the model will take us **remarkably far in understanding the process of economic growth** and the sources of long-run prosperity. There will be much more to say after this lecture, but the basic Solow model makes a great contribution and is indeed an important **workhorse model** in economics.

The basic Solow model

■ The model presented in this lecture describes a **closed economy**. Initially we will **not** explicitly include the **public sector** in the model, but we will show that it can easily be interpreted as including government expenditure and taxation.

Agents, commodities and markets

■ The **economic agents** in the model are **households**, also referred to as consumers; **firms**, also called producers; and possibly a government. Time runs in a **discrete sequence of periods** indexed by t. A period should be thought of as **one year**. There are **three commodities** in each period, and for each commodity there is a market. The commodities are **output**, **capital services** and **labour services**.

■ In the market for output the **supply consists of the total output of firms**, Y_t. The **demand** is the **sum of total consumption**, C_t, and **total investment**, I_t. Hence **output can be used either for consumption, or it can be transformed into capital via investment** (at no cost, we assume).

■ The model is thus a **one-sector model** and **does not distinguish between production of consumption goods and production of capital goods**. The **price** in the output market is **normalized to one**, so other **prices are measured in units of output**.

■ We may think of the accumulated stock of physical **capital** as being **directly owned by the households who lease it to the firms**. Hence, in the market for capital services the supply comes from the consumers. By a convenient definition of the unit of **measurement for capital services (machine years)** a capital stock of K_t units of capital can give rise to a supply of K_t units of capital services during period t. The **demand for capital services**, K_t^d, **comes from the firms**. The **real price**, r_t, in this market is the **amount of output that a firm must pay to a consumer for leasing one unit of capital** during period t. Thus r_t is a **real rental rate**.

■ This **rental rate has a close association with, but is not equal to, the model's real interest rate**. The reason is that **capital depreciates**. We assume that the use of one unit of physical capital for one period implies that an amount δ, where $0 < \delta < 1$, must be set aside to compensate for depreciation, that is, for the capital that is worn out by one period's use. The name for δ is the rate of depreciation. The model's **real interest rate**, ρ_t, is the **return to capital net of depreciation**, that is, $\rho_t = r_t - \delta$. **This is the rate of return on capital comparable to an interest rate earned from a financial asset like a bond**.

■ **Alternatively,** one can **think of physical capital as being owned by firms who finance their acquisition of capital by issuing debt to consumers**. In the latter case the **real price of the use of a unit of capital** for one period, also called the **"user cost"**, is the **real interest rate on debt plus the depreciation on one unit of capital**. In this interpretation one can think of the firm's **"rental rate"** as $r_t = \rho_t + \delta$. For our purposes it does not matter whether the user cost of capital, r_t, is a direct leasing rate or the sum of an interest rate and a depreciation rate.

■ In the labour market the **supply of labour services**, L_t, **comes from the households**, while the **demand**, L_t^d, comes **from the firms**. We measure labour flows in **man years**, so a labour force of L_t can give rise to a labour supply of L_t. The **real wage rate** in period t is denoted by w_t.

■ All **three markets are assumed to be perfectly competitive**, so economic **agents take the prices as given**, and in each market the appropriate p**rice adjusts so that price taking supply becomes equal to price taking demand**. This implies that available **resources are**

fully utilized in all periods or, in an **alternative interpretation**, that they are **utilized up to the "natural rate"**.

The production sector

■ The production side of the economy is modelled as if all **production takes place in one representative profit-maximizing firm**. The firm uses capital input, K_t^d, and labour input, L_t^d, to produce output (value added), Y_t, according to the production function:

$$Y_t = F(K_t^d, L_t^d) \qquad 3.1$$

■ The **production function** is assumed to display **constant returns to scale**. In mathematical terms this means that the function $F(K_t^d, L_t^d)$ is **homogeneous of degree one**, so $F(\lambda K^d, \lambda L^d) = \lambda F(K^d, L^d)$ for all $\lambda > 0$. In words, if we increase both inputs by, say, 100 per cent, output will also go up by 100 per cent. There is a **replication argument** for this assumption: it should be possible to produce twice as much from a doubling of inputs, since one could simply apply the **same production process twice**.

■ The **marginal product of capital** is the **increase in output generated by an extra unit of capital**. Formally, it is the **partial derivative** $F_K(K^d, L^d)$ of $F(K^d, L^d)$ **with respect to** $\boldsymbol{K^d}$. Similarly, the marginal product of labour is the partial derivative of the production function with respect to L^d, denoted by $F_L(K^d, L^d)$. We assume that $\boldsymbol{F_K}$ **and** $\boldsymbol{F_L}$ **are positive for all input combinations** (K^d, L^d).

■ In accordance with standard microeconomic theory, the **marginal products are also assumed to be decreasing in the amount of the factor used**, so $F_{KK}(K^d, L^d) < 0$ and $F_{LL}(K^d, L^d) < 0$, where F_{KK} is the **second derivative** with respect to K^d, and F_{LL} is the second derivative with respect to L^d. Finally, **when the input of one factor increases, the marginal product of the other factor is assumed to go up**. This means that $F_{KL}(K^d,L^d) = F_{LK}(K^d,L^d) > 0$.

■ The variables K_t^d and L_t^d are the amounts of capital and labour demanded by the firm in period t. The **firm faces the real factor prices** r_t and w_t, and it will want to choose Y_t, K_t^d and L_t^d to **maximize pure profits** $Y_t - rK_t^d - wL_t^d$, subject to the technical feasibility constraint, $Y_t = F(K_t^d, L_t^d)$. From microeconomic theory we know that a **competitive profit-maximizing firm will want to employ a factor of production up to the point where its marginal product is just equal to its real price**. In our model, the necessary first order conditions (resulting from differentiation, etc.) take the form:

$$F_K(K_t^d, L_t^d) = r_t \quad 3.2$$

$$F_L(K_t^d, L_t^d) = w_t \quad 3.3$$

■ Note that **(3.2) and (3.3) do not determine the levels of K_t^d and L_t^d given r_t and w_t**, even though they are **two equations in two unknowns**. If one combination (K_t^d, L_t^d) fits in the two equations, so does $(\lambda K^d, \lambda L^d)$ for any $\lambda > 0$. This is a **consequence of the assumed homogeneity** of the production function. If a function is homogeneous of degree k, then its first partial derivatives are homogeneous of degree $k - 1$. Since F is homogeneous of degree one, F_K and F_L are homogeneous of degree zero. Hence, $F_K(\lambda K^d, \lambda L^d) = F_K(\lambda K^d, \lambda L^d)$.

■ Nevertheless, an **optimal combination of factor inputs**, K_t^d and L_t^d, **must fulfil (3.2) and (3.3) to be optimal**.

■ Because the factor markets are perfectly competitive, the **rental rate**, r_t, **adjusts to equate capital demand with capital supply**, and the **wage rate**, w_t, **adjusts such that labour demand becomes equal to labour supply**. We will argue below that **as long as the rental rate is just slightly positive**, it will pay for **consumers** to **supply all of their capital to the market**, so capital supply is equal to the capital stock, K_t. Furthermore, we are going to assume that **consumers supply labour inelastically**, so labour supply must equal the labour force, L_t. Inserting $K_t^d = K_t$ and $L_t^d = L_t$ into (3.2) and (3.3), respectively, gives:

$$F_K(K_t, L_t) = r_t \quad 3.4$$
$$F_L(K_t, L_t) = w_t \quad 3.5$$

■ **In each period** t, the **stocks of capital and labour are predetermined and given**, as will be explained below. Therefore **(3.4) and (3.5) determine the period's rental rate**, r_t (and hence the interest rate, $r_t - \delta$), **and the wage rate**, w_t, respectively, as **these depend on the given levels of capital and labour**. This **leads to a theory of the functional distribution** of income.

The distribution of income and the Cobb-Douglas production function

■ Using (3.1), (3.4) and (3.5), we find that the shares of capital and labour in total income in any period t are:

$$\frac{r_t K_t}{Y_t} = \frac{F_K(K_t, L_t) K_t}{F(K_t, L_t)} \qquad 3.6$$

$$\frac{w_t K_t}{Y_t} = \frac{F_L(K_t, L_t) L_t}{F(K_t, L_t)} \qquad 3.7$$

■ Note that

$$\frac{F_K K}{Y} = \frac{\delta Y}{\delta K}\frac{K}{Y}$$

is the **elasticity of output with respect to the input of capital**, and

$$\frac{F_L L}{Y} = \frac{\delta Y}{\delta L}\frac{L}{Y}$$

is the **elasticity of output with respect to labour input**. Thus equations (3.6) and (3.7) say that the **share of each factor in total income is equal to the elasticity of output with respect to that factor**.

■ Notice also that the firm's pure profit measured in real terms is $Y_t - (r_tK_t + w_tL_t) = F(K_t, L_t) - [F_K(K_t, L_t)K_t + F_L(K_t, L_t)L_t]$. **Since the production function is homogeneous of degree one**, we have $F(K_t, L_t) = F_K(K_t, L_t)K_t + F_L(K_t, L_t)L_t$, **implying that pure profits are zero**.

■ This is an **appealing property of a perfectly competitive economy in long-run equilibrium**, because **if pure profits were positive**, we would **expect new firms to enter** the market until profits were competed away, and if pure profits were negative, we would expect some failing firms to be driven out of the market until the remaining firms were able to break even.

■ In Lecture 2 we saw that **in the long run the factor income shares are remarkably constant**. How does this constancy fit with our theory of income distribution?

■ **At first sight it does not seem to fit**. According to (3.7), the **labour income share is given as the output elasticity with respect to labour input, and this will generally depend on the input combination**, (K_t, L_t), **which is changing over time**. More precisely, labour's share depends on the capital-labour ratio, K_t/L_t. (Write labour's share as $F_L(K,L)/[F(K,L)/L]$. The numerator only depends on K/L, because F_L is homogeneous of degree zero, so $F_L(K,L) - L^0 F_L[K/L, 1) = F_L(K/L, 1)$, and the denominator only depends on K/L, because F is homogeneous of degree one, so $F(K, L)/L = LF(K/L, 1)/L = F(K/L, 1)$.

■ **Over time the capital-labour ratio has been systematically increasing in developed countries**. **Apparently this should have implied a changing labour income share**, according to our theory.

■ A similar **observation was already puzzling the economist Paul Douglas in 1927**. He asked the **mathematician Charles Cobb** whether a production function exists which has all the properties assumed above, and which will produce constant income shares when factors are paid their marginal products. As we have seen in (3.6) and (3.7), this is the same as requiring the **output elasticities with respect to capital and labour to be constant** (and hence independent of K_t/L_t). Charles Cobb found that there exists one such function, since then called the **Cobb-Douglas production function**:

$$Y_t = F(K_t^d, L_t^d) = B(K_t^d)^{\alpha}(L_t^d)^{1-\alpha},\ B > 0,\ 0 < \alpha < 1 \qquad 3.8$$

■ Here α and B are given parameters, and ***B* is often referred to as the total factor productivity**. Note that the **inputs of capital and labour are indexed by *t***, capturing that these amounts of **input may vary over time**.

■ The total factor productivity, ***B*, has no time subscript**, so it is assumed here that there is **no change in technology over time**. This is the sense in which the model in this lecture is the **basic Solow model**. Later we will consider the Solow model with technological progress where B is increasing over time.

■ Taking partial derivatives in (3.8), and inserting the equilibrium conditions, $K_t^d = K_t$ and $L_t^d = L_t$, we find that Eqs (3.4) and (3.5) take the particular form:

$$F_K(K_t, L_t) = \alpha B \left(\frac{K_t}{L_t} \right)^{\alpha - 1} = r_t \qquad 3.9$$

$$F_L(K_t, L_t) = (1-\alpha) B \left(\frac{K_t}{L_t} \right)^{\alpha} = w_t \qquad 3.10$$

■ Using (3.8), (3.9) and (3.10), you should check carefully that the **Cobb-Douglas production function** does indeed have all the properties attributed to *F*, and that it does **imply constant income shares**, a **capital share of *α***, and a **labour share of 1 – *α***.

■ The proximate constancy of income shares in the long run suggests that a production function of the **Cobb-Douglas form is a reasonable approximation for an aggregate production function for long-run analysis**. The fact that the **labour income share stays rather close to 2/3** (see Lecture 2) suggests that we have a reasonable value for the ***α*** appearing in (3.8), namely a number **around 1/3**.

■ **Both these features will be used extensively** in the lectures on the growth theory. Aggregate production functions will be assumed to have the Cobb-Douglas form, and whenever we need an estimate of α, we will consider 1/3 as a reasonable value.

The household sector

■ The **number of consumers** in period t is L_t, which is **predetermined from the past evolution of the population**. The **behaviour of consumers** is described by **four features**.

■ First, as mentioned, **each consumer is assumed to supply one unit of labour inelastically** in each period, so the total supply of labour is L_t. If households were **trading off consumption against leisure** so as to maximize utility, an **inelastic supply of labour would follow if the income and substitution effects from a change in the real wage were exactly offsetting each other**. Empirically, **long-run labour supply is not too sensitive to changes in real wages**, so the assumption of an inelastic labour supply is not a bad first approximation.

■ Second, the consumers own the **capital stock** K_t, which is **predetermined from capital accumulation in the past**. For any positive rental rate, $r_t > 0$, a **consumer seeking to maximize his income will want to lease all his capital to the firms**, because each consumer considers r_t as given. Therefore the **supply of capital services in year *t* is inelastic** and equal to the size of the capital stock K_t.

■ Third, each **consumer must decide how much to consume and how much to save**. We assume that the **household sector behaves as one representative consumer who earns all the economy's income**, Y_t.

■ The representative consumer must **decide on consumption**, C_t, and **hence on gross saving**, $S_t = Y_t - C_t$, respecting his **intertemporal budget constraint**. According to this constraint, the **addition**, $K_{t+1} - K_t$, **to the consumer's real stock of capital** from period t to $t + 1$ must **equal his gross saving** in period t **minus the depreciation**, δK_t, on the capital leased to the firms during period t. Hence the **intertemporal budget constraint is**:

$$K_{t+1} - K_t = S_t - \delta K_t \qquad 3.11$$

■ If we were **deriving consumption and savings behaviour from explicit optimization**, a **utility function** defined over streams of present and future consumption would be **maximized under the intertemporal budget constraint** (3.11).

■ Here, however, we make a **simplifying short cut**. We **assume that the outcome of the underlying optimization is a constant savings rate**, i.e., the household sector **saves the exogenous fraction**, s, of total income in each period:

$$S_t = sY_t, \text{ where } 0 < s < 1 \qquad 3.12$$

■ We are going to show below that the **assumption** of a given, **constant rate of saving and investment is empirically plausible in the long run**. The intertemporal budget constraint, (3.11), is itself important in our model: it is the aggregate capital accumulation equation stating that **new capital originates from net savings**.

■ Fourth, the **"biological" behaviour of the households** is described by an **exogenous growth rate, *n*, of the population**, or rather of the labour force:

$L_{t+1} = (1 + n)L_t$, where $n > -1$ 3.13

The complete model

■ The **basic Solow model** consists of the following **six equations**, all repeated from above:

$$Y_t = BK_t^{\alpha} L_t^{1-\alpha} \quad 3.14$$

$$F_K(K_t, L_t) = \alpha B\left(\frac{K_t}{L_t}\right)^{\alpha-1} = r_t \quad 3.15$$

$$F_L(K_t, L_t) = (1-\alpha)B\left(\frac{K_t}{L_t}\right)^{\alpha} = w_t \qquad 3.16$$

$$S_t = sY_t \qquad 3.17$$

$$K_{t+1} - K_t = S_t - \delta K_t \qquad 3.18$$

$$L_{t+1} = (1+n)L_t \qquad 3.19$$

■ The **parameters of the model** are α, B, s, n and δ. Given an **initial input** combination K_0, L_0 in some initial year zero, these **six equations determine the dynamic evolution of the economy**, as illustrated in Figure 3.1.

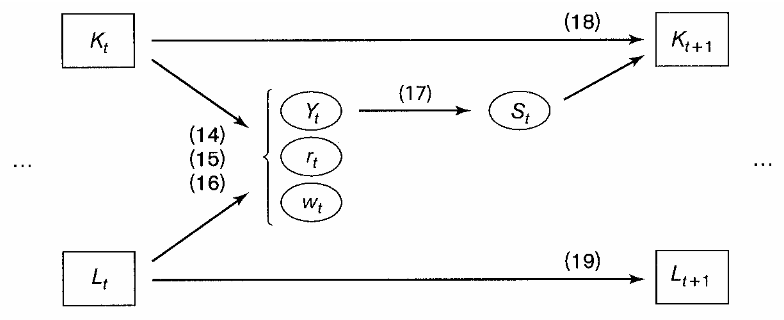


Figure 3.1: The dynamics of the basic Solow model

Note: Predetermined endogenous variables in squares, endogenous variables that can adjust during the period in circles.

■ In period t the inputs K_t and L_t are predetermined, so the first of the above equations gives the supply-determined output, Y_t, directly while the next two equations give the rental rate, r_t, and the wage rate, w_t, respectively. With Y_t determined, Eq. (3.17) gives the gross saving, S_t, in period t, and then from S_t and K_t, Eq. (3.18) determines the next period's capital supply,

K_{t+1}. Finally, from the labour supply L_t of period t, Eq. (3.19) determines the next period's labour supply, L_{t+1}. Hence, K_{t+1} and L_{t+1} have been determined, and one can start all over again with period $t + 1$ etc.

■ We have chosen to present the fundamental **capital accumulation equation**, (3.18), **as the representative consumer's intertemporal budget constraint**. We could instead have explained the capital accumulation equation in the following **more direct way**: by definition, **gross investment**, I_t **minus depreciation**, δK_t, **is the addition to capital**: $K_{t+1} - K_t = I_t - \delta K_t$. The **national accounting identity for the output market**, $Y_t = C_t + I_t$, is equivalent to $I_t = S_t$, since $S_t = Y_t - C_t$. Inserting S_t for I_t in the definition of gross investment gives (3.18).

*Government

■ The more direct way of viewing the capital accumulation equation is useful for extending the **model interpretation to include a government**. Assume that the **government collects a tax** revenue of T_t in period t, and that government expenditure for public consumption and public investment is C_t^g and I_t^g, respectively. Let **expenditure behaviour** be described by the

ratio of public consumption to GDP, c_t^g, and by the **ratio of public investment to GDP**, i_t^g, respectively:

$$C_t^g = c_t^g Y_t, \quad \text{where } 0 < c_t^g < 1,$$
$$I_t^g = i_t^g Y_t, \quad \text{where } 0 < i_t^g < 1.$$

■ We allow c_t^g and i_t^g to be time dependent and assume $c_t^g + i_t^g < 1$. Assume further that the **government balances its budget**:

$$T_t = C_t^g + I_t^g$$

■ Let c_t^p denote the ratio of private consumption, C_t^p, to private disposable income, $Y_t - T_t$. Then:

$$C_t^p = c_t^p (Y_t - T_t), \quad \text{where } 0 < c_t^p < 1,$$

where we also allow c_t^p to be time-dependent. Combining the above equations one finds that:

$C_t^p = c_t^p\ (1 - c_t^g\ \text{--}\ i_t^g)\ Y_t,$

■ Thus **private consumption as a fraction of GDP** is $c_t^p\ (1 - c_t^g - i_t^g)$, which is not necessarily a constant. **Total (national) consumption**, $C_t = C_t^g + C_t^p$, is:

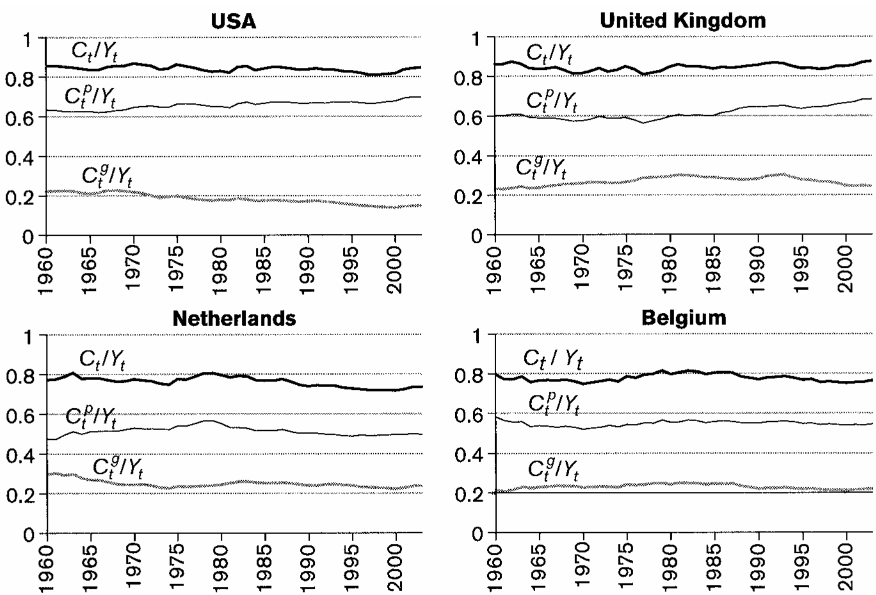
$C_t = [c_t^g + c_t^p\ (1 - c_t^g - i_t^g)]Y_t.$

■ The assumption we make is that the share of total consumption in GDP, $c_t^g + c_t^p\ (1 - c_t^g - i_t^g)$, is a given constant. Then national savings must be a constant fraction of GDP as well, $S_t = sY_t$, where:

$$s = 1 - c_t^g - c_t^p\ (1 - c_t^g - i_t^g) = (1 - c_t^p)(1 - c_t^g) + c_t^p\ i_t^g \qquad 3.20$$

■ Again, **national gross investment must equal national gross saving**, $I_t = S_t$. Hence, all we need to do to interpret a government into our model (3.14)-(3.19) is to think of the constant saving and investment rate, s, appearing in (3.17) as being given by underlying parameters as in (3.20). Note that s, the national investment rate, is decreasing in c_t^p and c_t^g, and increasing in i_t^g (affects s through the term $c_t^p\, i_t^g$).

■ An essential **assumption** underlying the Solow model is that the **sum of the GDP shares of private and public consumption**, which is the share of total consumption in GDP, is a given **constant**. Figure 3.2 indicates that the **ratio of total consumption to GDP has indeed stayed relatively constant over long periods in developed economies**, with a slight **tendency** for C_t^p/Y_t and C_t^g/Y_t **to move in opposite directions**, keeping the sum, C_t/Y_t, relatively stable.



USA
United Kingdom
Netherlands
Belgium
C_t/Y_t
C_t^p/Y_t
C_t^g/Y_t
1
0.8
0.6
0.4
0.2
0
1960
1965
1970
1975
1980
1985
1990
1995
2000

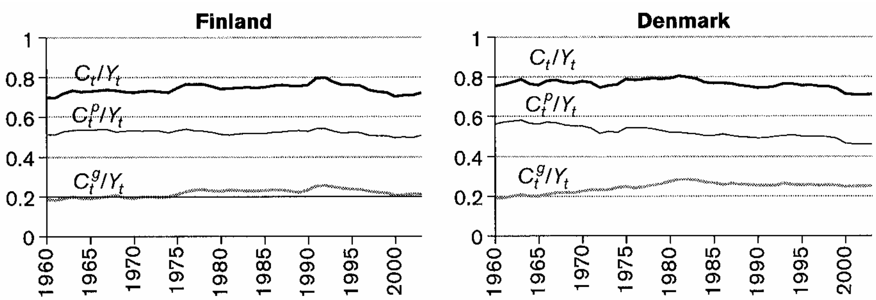


Figure 3.2: The consumption share of GDP in several Western countries

Source: OECD Economic Outlook database.

■ For a typical Western developed country, a **reasonable value for the model parameter *s* and investment rates are about 20 per cent**, while less developed countries can have investment rates far below 20 per cent.

■ Sometimes breaks in national saving and investment rates occur. **We are not arguing that savings rates stay constant forever**, only that they tend to stay **sufficiently constant over sufficiently long periods** to justify a modelling **assumption of an exogenously given savings and investment rate**. Indeed, one **important economic event** that we want to be able to analyse in our growth model is a **permanent structural change in the savings rate** from one constant value to another.

Money

■ There is **no direct trace of money in the Solow model**. This is because most economists think that **in the long run money does not matter for real economic variables**. One should treat this statement with caution. Certainly the **quality of the financial system can have a long-run impact**, for instance by **affecting our parameter *s***.

■ Likewise, many economists think that the **quality of monetary stabilization policy**, that is, the systematic way that monetary authorities respond to economic events, **can have economic consequences also in the long run**.

■ However, given the quality of the financial system, most economists think that the exact level or **growth rate of the nominal money stock**, or of the **short-term nominal interest** rate set by the central bank, **do not affect real variables such as GDP in the long run**.

■ Figure 3.3 gives a sharp illustration. For example, in the US a **change in money growth of a given size goes hand in hand with a change in inflation** of the same size in the longer run (note that the figure focuses on the long run by considering averages over periods of 10 years).

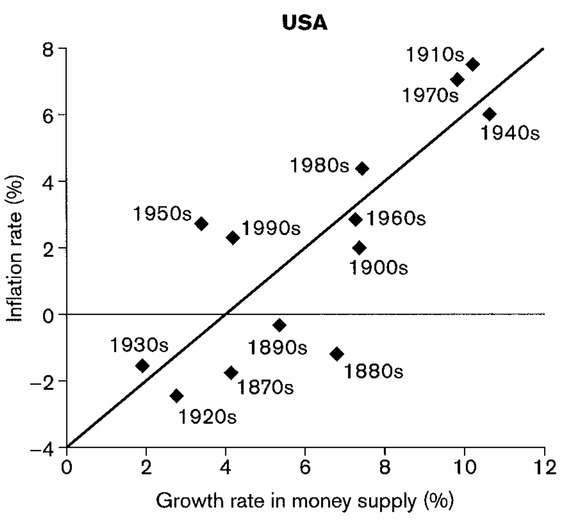


Figure 3.3: Money growth and inflation in the US, 1870-2000

Note: Money growth and inflation rates are average annual rates over each decade. Money supply is measured by M2.
Sources: M. Friedman and A. Schwartz, *A Monetary History of the United States 1867-1960*, The University of Chicago Press, 1982 and IMF International Financial Statistics.

■ The **figure does not prove that money growth has no real impact in the long run**. There is, however, a rather sophisticated econometric literature on this issue. Carl E. Walsh (*Monetary Theory and Policy*, Cambridge and London, MIT Press, 2000) gives an account of this literature and summarizes as follows as regards the long run: "**Money growth and inflation display a correlation of 1**; the **correlation between money growth** or inflation **and real output growth is probably close to zero**".

■ On the other hand, the conclusion on the **short-run effects of money** is that "**exogenous monetary policy shocks produce hump-shaped movements in real economic activity**. The **peak effects occur after a lag of several quarters** (as much as two or three years in some of the estimates) and then die out".

■ This should explain why **you will hear no more of money in the economic growth part** of these lectures, while **monetary policy will be at the heart of the analysis of business cycles**.

Analysing the basic Solow model

Output and capital per worker

■ For the prosperity of a nation it is **GDP per worker** or per capita, not GDP itself, **that is important**. In the Solow model we are therefore interested in **output per worker**, $y_t \equiv Y_t/L_t$.

■ Define similarly capital per worker in period t, also called the capital-labour ratio, or the **capital intensity**: $k_t \equiv K_t/L_t$. From the first equation of the Solow model, (3.14) above, it follows from **dividing on both sides by L_t** that:

$$y_t = Bk_t^{\alpha} \qquad 3.21$$

■ This is like a **production function where output is GDP per worker and input is capital per worker**. This production function exhibits **diminishing returns with the marginal product**, $\alpha Bk_t^{\alpha-1}$, **decreasing from infinity to zero as k_t increases from zero to infinity**. The per capita production function is illustrated in Figure 3.4.

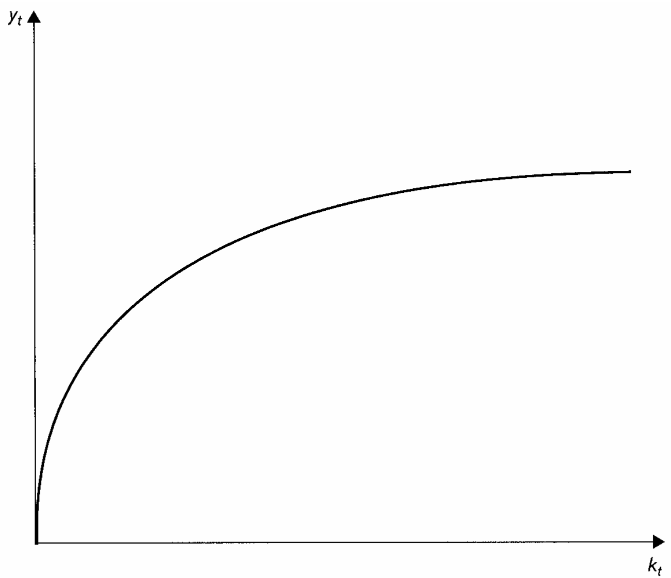


Figure 3.4: The per capita production function

■ One feature revealed directly by (3.21) is that in the basic Solow model (where B is assumed to be constant), an **increase in production per worker can only come from an increase in capital per worker**.

■ Let the approximate growth rates of y_t and k_t from period $t – 1$ to period t be denoted by g_t^y and g_t^k, respectively. Writing an equation like (3.21) for both period t and period $t – 1$, taking **logs on both sides**, and **subtracting** gives:

$$g_t^y \equiv \ln y_{t+1} - \ln y_t = \alpha(\ln k_{t+1} - \ln k_t) \equiv \alpha g_t^k \qquad 3.22$$

■ According to (3.22), the **(approximate) growth rate in output per worker is proportional to the growth rate in capital per worker**, and the **proportionality factor is exactly the capital share**, α. In the basic Solow model **economic growth is thus completely linked to capital accumulation**.

The law of motion

■ We can **analyse the Solow model directly in terms** of the variables we are interested in, the **per capita (or worker) magnitudes**, k_t and y_t. First, insert the savings behaviour (3.17) into the capital accumulation equation (3.18) to get:

$$K_{t+1} = sY_t + (1 - \delta)K_t$$

■ Then **divide this equation on both sides by L_{t+1}**, on the right-hand side using that $L_{t+1} = (1 + n)L_t$. Using also the definitions of k_t, and y_t, one gets:

$$k_{t+1} = \frac{1}{1+n}[sy_t + (1-\delta)k_t]$$

■ Finally, from the per capita production function, (3.21), we can substitute Bk^α for y_t to arrive at:

$$k_{t+1} = \frac{1}{1+n}[sBk_t^{\alpha} + (1-\delta)k_t] \qquad 3.23$$

■ This is the **basic law of motion**, or the **transition equation**, **for the capital intensity** k_t following from the basic Solow model. Note that (3.23) has the form of a first-order, one-dimensional, non-linear **difference equation**. For a given initial value, k_0, of the capital-labour ratio in year zero, (3.23) determines the capital-labour ratio, k_1 of year one, which can then be inserted on the right-hand side to determine k_2, etc. In this way, given k_0, the transition equation (3.23) determines the **full dynamic sequence (k_t) of capital intensities**. The **notation (k_t)** will be used for the full dynamic sequence for k_t that is, (k_t) lists k_t, for t = 0, 1, 2, ...

■ The **four equations** (3.14) and (3.17)-(3.19), which gave the dynamic evolution of Y_t, S_t, K_t and L_t, have thus been **boiled down to the single equation** (3.23) giving the dynamic evolution of k_t. With (k_t) determined, the **rental rates will be given from (3.15) and (3.16)**:

$r_t = \alpha B k_t^{\alpha\text{-}1}$ 3.24

$w_t = (1 - \alpha) B k_t^{\alpha}$ 3.25

■ The **sequence for output per worker**, (y_t), follows from (k_t) by $y_t = Bk_t^{\alpha}$. Total consumption in period t is $C_t = Y_t - S_t = (1 - s)Y_t$, so consumption per worker is $c_t = (1 - s)y_t$. Therefore (c_t) follows from (k_t) via $c_t = (1 - s)Bk_t^{\alpha}$. In this way the **dynamic evolution of all the variables of interest can be derived, given an initial capital intensity**.

■ There is **another illustrative way to state the economy's law of motion**, where one expresses the change in the capital intensity as a function of the current capital intensity. Subtracting k_t from both sides in (**3.23**) gives the so-called **Solow equation**:

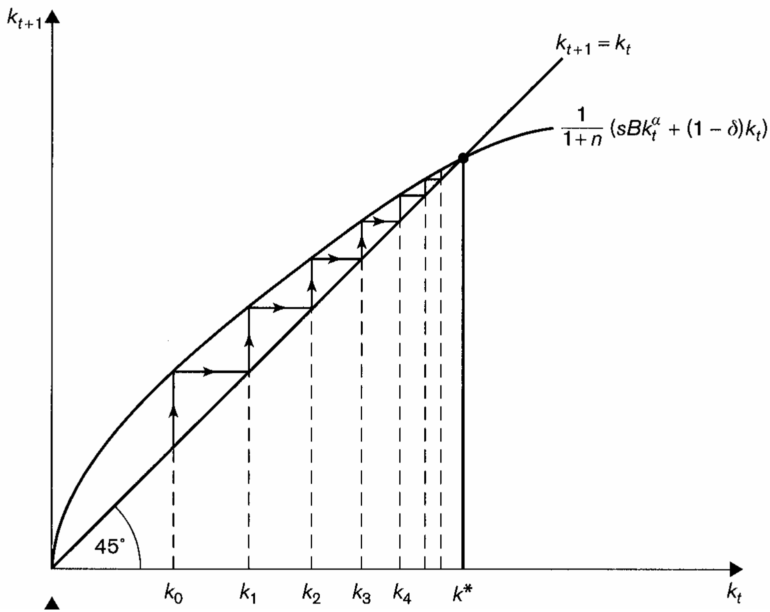
$$k_{t+1} - k_t = \frac{1}{1+n}[sBk_t^{\alpha} - (n - \delta)k_t] \qquad 3.26$$

■ This has a nice intuitive interpretation. On the **left-hand side** is the **increase in capital per worker**. On the right-hand side are the **elements that can create such an increase**: **saving per worker** in period t is $sy_t = sBk_t^{\alpha}$, which **adds** to capital per worker, **depreciation per worker** in period t is δk_t, which **subtracts** from capital per worker, and finally there is a **subtraction**, nk_t, stemming from the fact that there are **more workers** in period $t + 1$ to share a given capital stock (if $n > 0$).

■ You may wonder why n enters (3.26) exactly as it does. Here is an explanation. **Assume that** in period t **saving and investment have a size that keeps the capital stock exactly unchanged** from t to $t + 1$: $K_{t+1} = K_t$. This means that **capital per worker will be reduced** from t to $t + 1$ by an amount caused **by population growth**, and **by that alone**. Using the definition of k_t, we have in that case: $k_{t+1}/k_t = (K_{t+1}/K_t/(L_{t+1}/L_t) = 1/(1 + n)$, so $k_{t+1} = k_t/(1 + n)$, and then $k_{t+1} - k_t = -[n/(1 + n)]k_t$. This shows how capital per worker evolves when there is no compensation for population growth. You should now recognize the way n enters (3.26).

Convergence to steady state

- Figure 3.5 illustrates the dynamics of the Solow model.


k_{t+1}
$k_{t+1} = k_t$
$\frac{1}{1+n}(sBk_t^{\alpha} + (1-\delta)k_t)$
45°
k_0
k_1
k_2
k_3
k_4
k^*
k_t

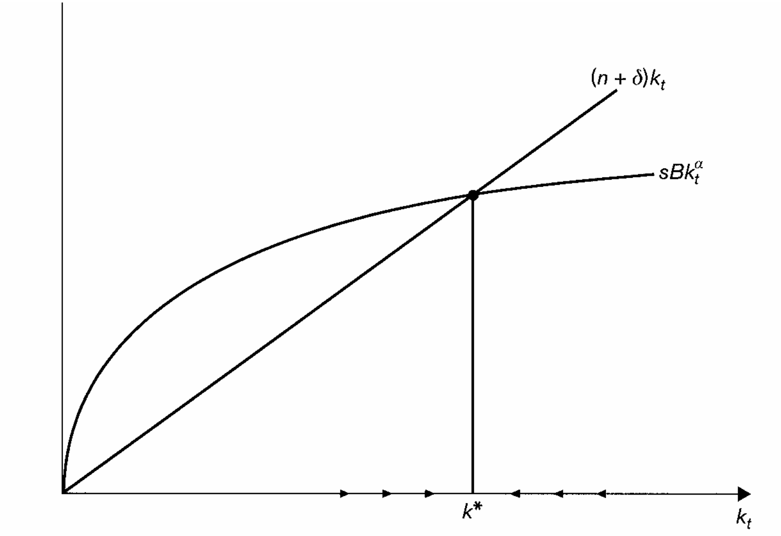


Figure 3.5: The transition diagram (top), and the Solow diagram (bottom)

■ The **upper part of the figure, the transition diagram**, shows the transition equation as given by (3.23). This curve starts at (0, 0) and is everywhere increasing. The **45° line**, $k_{t+1} = k_t$, has also been drawn. Differentiating (3.23) gives:

$$\frac{dk_{t+1}}{dk_t} = \frac{s\alpha Bk_t^{\alpha-1} + (1-\delta)}{1+n}$$

■ This shows that the **slope of the transition curve decreases monotonically from infinity** to $(1 - \delta)/(1 + n)$, as k_t, increases from zero to infinity. The latter slope is positive and less than one if $n + \delta > 0$, or $n > -\delta$, which is plausible empirically.

■ Usually **annual depreciation rates** on aggregate capital are estimated to be **between 5 and 15 per cent** (there is some uncertainty here because there are **many kinds of capital** with different depreciation rates), while most often the lower end of the interval around 5 per cent is considered most plausible for aggregate capital. **Population growth usually rates never slump to –0.05**. We can therefore **safely assume** $n > -\delta$, implying that the **slope of the**

transition curve falls monotonically from infinity down to a value less than one. Hence the **transition curve must have a unique intersection with the 45° line** (which has **slope one**), to the right of $k_t = 0$.

■ The **lower part** of Figure 3.5, the so-called **Solow diagram**, illustrates the Solow equation (3.26). It contains the **curve** sBk_t^{α}, and the **ray** $(n + d)k_t$. The **vertical distance between the first and the second**, divided by $(1 + n)$, is $k_{t+1} - k_t$ according to (3.26).

■ The two curves must **intersect each other once to the right** of $k_t = 0$, since the **ray has constant slope** $n + \delta > 0$, and the **slope of the curve falls monotonically from infinity to zero**. **Where this intersection occurs, we have $k_{t+1} - k_t = 0$, or $k_{t+1} = k_t$.** Hence the intersection must be at the same k_t at which the transition curve intersects the 45° line in the upper part of the figure.

■ The iterations of (3.23) described above appear in the transition diagram as follows. Assume an initial capital intensity $k_0 > 0$, as drawn. Then k_1 will be the vertical distance from the horizontal axis up to the transition curve, and by going from the associated point on the transition curve horizontally to the 45° line, and then vertically down, k_1 will be taken to the horizontal axis as shown. Now, k_2 will be the vertical distance up to the transition curve. The **dynamic evolution of the capital intensity is given by the staircase-formed broken line**.

■ It follows that **over time, k_t will converge to the specific value** given by the intersection between the transition curve and the 45° line, and it will do so **monotonically**, getting closer and closer all the time and **never transcending to the other side of the intersection point**. Furthermore, the **convergence is global** in the sense that it holds for any strictly positive initial k_0.

■ In the Solow diagram, **when the curve is above the ray**, one must have $k_{t+1} - k_t > 0$, or $k_{t+1} > k_t$, so **k_t must be increasing over the periods**, and **vice versa**. To the left of the intersection point, k_t is therefore increasing over time, while **to the right it is decreasing**, as indicated by the **arrows along the k_t axis**.

■ In **conclusion**, the dynamics of the Solow model are such that from any strictly positive initial value, k_0, the **capital intensity will converge monotonically to a specific value** given by the intersection points in Figure 3.5. This value is the unique positive solution, k^*, which is obtained by setting $k_t = k_{t+1} = k$ in (3.23) or (3.26) and solving for k. This k^* is called the **steady state value for the capital intensity**. There is an **associated steady state value for output per worker**, $y^* = B(k^*)^\alpha$, and y_t converges to y^* over time.

Comparative analysis in the Solow diagram

■ Figure 3.6 is a Solow diagram just like the lower part of Figure 3.5.

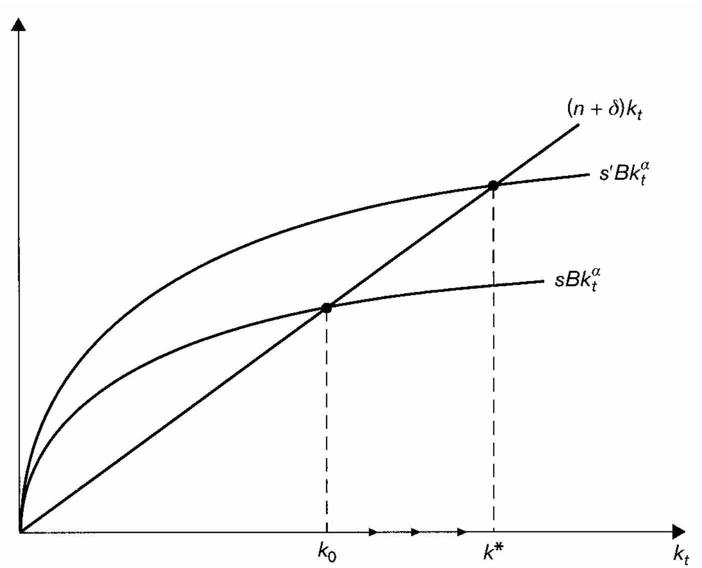


Figure 3.6: An increase in the savings rate in the Solow diagram

■ It contains the ray $(n + \delta)k_t$ and two different curves, sBk_t^{α} and $s'Bk_t^{\alpha}$, where $s' > s$. We imagine that the economy has first been characterized by the parameter values α, B, s, n and δ for a long time, but then **from some year zero, the gross savings rate shifts** to s' and **stays at its new higher level thereafter**, while no other parameters change.

■ Remember that since $s = S_t/Y_t = I_t/Y_t$ we may as well say that it is the **gross investment rate that has increased**. Since the old parameter values have prevailed for a long time, the **economy should initially be at (or close to) the steady state** corresponding to the old parameter values, k_0 in the figure. **In this steady state current savings are just large enough to compensate capital per worker for depreciation and population growth**, so k_t stays unchanged from period to period.

■ **In year zero, when the increase in the savings rate occurs, the initial capital intensity will be unchanged**, k_0, because it is predetermined and given by capital accumulation and population dynamics in the past, where the savings rate was s. Therefore output per worker stays unchanged at $y_0 = Bk_0^{\alpha}$ as well. **In the short run, output and income per worker are**

thus unaffected. However, out of the unchanged income per worker, y_0 a larger part will now be saved, $s'y_0$ in period zero against sy_0 previously. **Current savings will therefore more than compensate for depreciation and population growth**.

■ In the figure, $s'Bk_0^{\alpha} > (n + \delta)k_0$, so **capital per worker will increase from period zero** to period one. With $k_1 > k_0$, one will also have $y_1 > y_0$, and out of the increased income per worker will come even more savings per worker, $s'Bk_1^{\alpha} > s'Bk_0^{\alpha}$. Hence capital per worker will increase again, $k_2 > k_1$ and so on. **In the long run successive increases in the capital intensity and in output per worker will make k_t converge to the new and higher steady state** value k^*, and y_t will converge to the new and higher steady state value $y^* = B(k^*)^{\alpha}$. These increases reflect the economy's increased capacity for saving.

■ One can **use the Solow diagram for another type of comparative analysis which does not involve a parameter shift**. As above, assume that the economy is initially in steady state at parameters α, B, s, n and δ, so the capital intensity is k_0, etc. **Suppose that the economy's capital per worker then drops** to 1/3 of its old value at the beginning of period one, say, **because most of the capital is destroyed in a war** or an earthquake during period zero, while the parameters stay unchanged. (Think of **Lithuania in the beginning of the transition** in the 1990s.)

■ We can analyse this situation in the **Solow diagram** by keeping the **curves in it unchanged**, but reducing the capital intensity to $(1/3)k_0$ from period one. You should do this in a Solow diagram and convince yourself that the **economy will immediately start a recovery back towards its initial steady state**. Perhaps this recovery can partly describe the **strong growth performance of some countries after the Second World War**.

Steady state

■ Our **model predicts that in the long run the capital intensity and GDP per worker converge to particular steady state levels**, k^* and y^*, respectively. It is of interest to express k^*, y^* and other key variables of the steady state as functions of the model's parameters α, B, s, n and δ. This will show us **which fundamental characteristics of an economy can create a high level of income** and consumption per worker in the long run according to the basic

Solow model. Furthermore, it will enable us to confront the model's steady state predictions with the data.

The key endogenous variables in steady state

■ The **steady state capital-labour ratio is given as the unique constant solution**, $k_{t+1} = k_t = k$, to (3.23) or (3.26) above. From the latter, such a k must fulfil: $sBk^{\alpha} - (n + \delta)k = 0$, or $k^{1-\alpha} = sB/(n + \delta)$. Denoting the solution by k^*, this gives:

$$k^* = B^{1/(1-\alpha)}\left(\frac{s}{n+\delta}\right)^{1/(1-\alpha)} \qquad 3.27$$

■ Given the assumption $n + \delta > 0$, this is a meaningful expression. The **steady state output per worker** is found by inserting the particular value k^* for k_t in the per capita production function, $y_t = Bk_t^{\alpha}$:

$$y^* = B^{1/(1-\alpha)}\left(\frac{s}{n+\delta}\right)^{\alpha/(1-\alpha)} \qquad 3.28$$

■ **Consumption per worker** is $c_t = (1 - s)y_t$ in any period t, so:

$$c^* = B^{1/(1-\alpha)}(1-s)\left(\frac{s}{n+\delta}\right)^{\alpha/(1-\alpha)} \qquad 3.29$$

■ Likewise, inserting k^* into (3.24) and (3.25) gives the **real factor prices in steady state**:

$$r^* = \alpha\left(\frac{s}{n+\delta}\right)^{-1} \qquad 3.30$$

$$w^* = (1-\alpha)B^{1/(1-\alpha)}\left(\frac{s}{n+\delta}\right)^{\alpha/(1-\alpha)} \qquad 3.31$$

■ These simple formulas contain some very sharp and important predictions about an economy's long-run performance. For instance, (3.28) gives a handy formula for **how an economy's GDP per worker should depend on a few basic structural parameters**, such as the rate of **investment** and the **population growth** rate, in the long run. Taking **logs** on both sides of (3.28) gives:

$$\ln y^* = \frac{1}{1-\alpha}\ln B + \frac{\alpha}{1-\alpha}[\ln s - \ln(n+\delta)] \qquad 3.32$$

from which, for instance, the **elasticity of y^* with respect to s is $\alpha/(1-\alpha)$** (you can show this by differentiating with respect to s). **Since α is around 1/3, the elasticity, $\alpha/(1-\alpha)$, should be around 1/2**. In other words, according to our model a **20 per cent increase in the**

investment rate, e.g. from 20 to 24 per cent, will imply an increase in people's average incomes of around 10 per cent in the long run.

■ **What can make a country rich in the long run?** The answer suggested by (3.28) is that a relatively **high technological level**, B, a **high rate of gross savings and investment**, s, a **low population growth rate**, n, and a **low depreciation rate**, δ, tend to imply relatively high GDP per worker in the long run. The importance of such a statement for matters of **structural economic policy is obvious**.

The "natural" interest rate: productivity and thrift

■ Just as (3.28) gives a prediction for the long-run GDP per worker, (3.30) gives a **prediction for the real interest rate**. Recall that the r^* of (3.30) is the steady state value for the rental rate for capital, while the **steady state real interest rate** is $\rho^* = r^* - \delta$, so:

$$\rho^* = \alpha \frac{n+\delta}{s} - \delta \qquad 3.33$$

■ The ρ^* of (3.33) is sometimes referred to as the **"natural" rate of interest**, following the Swedish economist **Knut Wicksell** who was **the first to use this term**. Knut Wicksell wrote at the turn of the twentieth century. Today he is considered as **one of the great figures in the history of economic thought**. His theory of interest rates is summarized in: Knut Wicksell, "The Influence of the Rate of Interest on Prices", *Economic Journal*, 17, 1907.

■ We see that the **natural rate of interest** in the basic Solow model is **determined by the forces of productivity and thrift**. The parameter α measures the response of output to an increase in capital input. **The higher the value of α, the higher is the productivity of capital**, and the **greater will be the demand for capital**, and hence **for replacement investment to compensate for depreciation and population growth in steady state**.

■ A higher value of $n + \delta$ increases the **demand for replacement investment** as well, and hence increases the equilibrium real interest rate, the price of capital. This is the influence of productivity on ρ^*. The influence of thrift is reflected in the appearance of s: **the higher the national propensity to save, the greater the supply of capital**, and the **lower the equilibrium real interest rate**, *ceteris paribus*.

■ As argued above, reasonable values by western standards for the parameters entering into (3.33) could, on an annual basis, be: $\alpha = 1/3$, $s = 0.22$, $n = 0.005$, and $\delta = 0.05$. These parameters result in a value for the natural interest rate per annum of $\boldsymbol{\rho^*}$ **= 3.3** per cent. This is very **close to the long-run average values of annual real interest rates implied by the observed historical difference between nominal interest rates on long-term bonds and the observed rates of inflation**.

■ Although we are going to revise our formula for the natural real interest rate somewhat in more sophisticated models later on, it is promising that our extremely simple model predicts a long-run real interest rate that is so close to empirical estimates.

Testing the model's steady state prediction of GDP per worker

■ We can **test our prediction of long-run GDP per worker empirically**, for instance by considering **cross-country data**. Here we run into the difficulty that it is **not so easy to get reliable data for differences between countries in technological variables** such as B and δ. However, we do have good data for GDP per worker and for the investment and population growth rates for many of the countries in the world. How well does the prediction of (3.28) fit with the data on these variables?

■ Figure 3.7 plots the **GDP per worker in 2000 against the average gross investment** rate between 1960 and 2000 across all countries for which data were available from the PWT and which got a grade for data quality higher than D in the PWT.

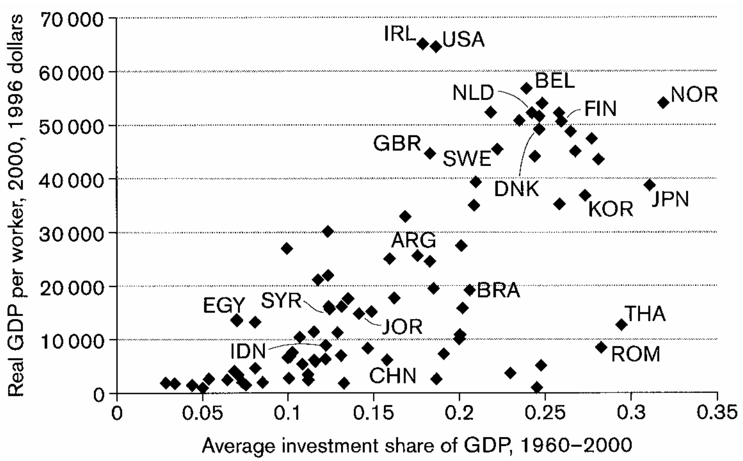


Figure 3.7: Real GDP per worker against the average investment share, 85 countries

Source: Perm World Table 6.1

■ Figure 3.8 plots **GDP per worker against average population growth** rates between 1960 and 2000 for the same countries.

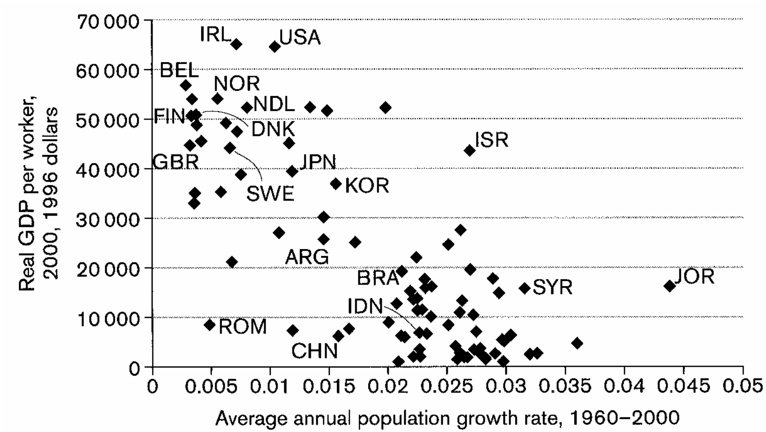


Figure 3.8: Real GDP per worker against the average annual population growth rate, 85 countries

Source: Penn World Table 6.1

■ The **increasing relationship in Figure 3.7** and the d**ecreasing one in Figure 3.8** are nicely **in accordance with** the basic **Solow model**. A **high savings rate and a low population growth** rate do seem to go hand in hand with a **high level of GDP per worker**.

■ Later we will discuss the goodness of these fits further. For now, the accordance between the data and the steady state prediction of the basic Solow model is promising and the **model takes us remarkably far in understanding the sources of long-run prosperity**.
|

Structural policy and the golden rule

Crowding out

■ Consider the **model interpretation with a government**, where the national savings and investment rate, s, is given by (3.20). **Assume that government increases its propensity to consume**, c_t^g, such that ***s* decreases permanently to a new and lower constant level** as given by (3.20). What will be the effects of such a policy according to our model starting from a steady state?

■ This is easy to answer since **we have already studied the effects of an increase in *s***, and the **effects are now just the opposite**. Hence, in the short run (in the first period of the new and lower *s*), where K_t is predetermined by capital accumulation in the past according to the old investment rate, both K_t and L_t, and hence Y_t, will be unaffected by the policy change: Y_t will continue to grow at the rate *n*, ensuring that GDP per worker stays unchanged in the first period.

■ This should remind you of lessons learned earlier, for example, in connection with the so-called **classical macroeconomic model**: in a period where **output is determined from the supply side by the available amounts of resources**, an **increase in government demand has no influence on GDP**, but **private investment will be fully crowded out** by the increase in government consumption.

■ **In the long run, the policy change** considered **will have an effect on GDP** and on GDP per capita. Out of the (unchanged) output and income in the first period of the new policy, a **smaller fraction than before will be saved and invested**, and therefore there will be less capital, and hence output, in the succeeding period, leading to a further fall in saving and capital accumulation, etc.

■ Over time Y_t **will thus develop more slowly than without the policy change**, thus bringing output per worker to a new and lower steady state value, **reflecting the economy's decreased potential for saving and investing**. In the long run there is **more than full crowding out**, a result arising from the explicit consideration of the capital accumulation process. The **decrease in capital supply caused by the government's increased propensity to consume** also implies that the **long-run equilibrium interest rate increases**, as we noted in connection with (3.33) above.

■ It is an important and **general insight that a government cannot create a positive influence on GDP or GDP per capita in the long run by boosting government consumption**.

■ However, **it may be able to do so in the short run if the economy is not currently at the capacity constraint**. In that case **a rise in public consumption will lead to less than full crowding out of private spending**, as we shall see in the lectures on short-run economic fluctuations and stabilization policies. Here we are concerned with the long run and with structural policies.

■ **Should one conclude from the above analysis that in the long run government expenditure can have only a negative influence** on economic activity and consumer welfare, because of its negative influence on capital accumulation? **Certainly not**, and for a number of reasons.

Government expenditure motivated by long-run concerns

■ First, instead of increasing its spending on consumption, the **government could choose to spend an increase in tax revenues on investment** in, say, **infrastructure**. In our model this possibility is represented by an increase in i_t^g rather than in c_t^g, and an increase in i_t^g has a positive influence on *s*, as shown by (3.20). Thus, a **tax financed increase in government's propensity to invest raises the national investment rate** to a new and higher level, which has a positive long-run impact on GDP per worker.

■ Second, **consumers may be willing to accept a fall in aggregate national income** arising from an increase in government consumption **if this increase enables the public sector to provide essential public goods** which cannot be provided in adequate quantities by private, profit-motivated producers via the marketplace.

■ Third, **if public consumption spent on, say, education and health is a near-perfect substitute for private spending on these items**, a tax-financed increase in **public spending on schools and hospitals will most likely reduce private consumption on education and health by a roughly similar amount**, leaving total saving and investment and thereby total output, unaffected.

■ Fourth, there may be **distributional reasons** for wanting part of, for example, health services to be publicly provided or financed. For educational services, both distributional concerns about equality of opportunity and the presence of **positive externalities** (e.g. the benefits to other citizens that a person is literate) **may motivate public provision or financing**.

■ Finally, much of what is normally categorized as **public consumption may be seen as delivery of productive inputs to the private sector production process**. For example, **health and education services may improve the productivity of the labour force**, and public institutions guaranteeing **law and order** (examples of essential public goods) may also be **productivity-enhancing** and encourage a **higher private savings rate by safeguarding property rights**. Thus part of public consumption may help to increase the productivity variable B as well as the private savings rate $1 - c_t^p$ in the Solow model.

Incentive policies

■ We have **considered policies that influence the national savings rate**, s, **through the government's direct control over public consumption and investment**. In addition, we can imagine that the **government can influence the private savings rate, or the population**

growth rate, **through incentive policies**, for instance policies that restructure the **tax system**. However, **our model does not give any explicit description of the underlying incentive mechanisms**.

■ Whether government is assumed to influence the model's basic parameters one way or the other, a main conclusion is that **a government wishing to promote its citizens' long-run incomes should** follow structural policies that can somehow **improve technology**, increase the national propensity to **save and invest**, and reduce **population growth**.

■ Certainly many countries use various programmes to enhance technology, and such activities may be meaningful according to the Solow model. However, **if some technology is only adopted by private firms because it is subsidized**, there should, from an economist's point of view, be a **positive external effect** associated with the technology, that is, a positive **spillover effect** on agents other than the firm investing in the technology.

■ **Policies to promote private savings** can be of many types. Quite often these policies take the form of **tax incentives** for savings and investment. As mentioned, we cannot conduct an explicit analysis of such incentive policies, since **we have not derived private savings behaviour from optimization**. We can, however, point to some **main issues**. For example, **is capital income overtaxed compared to labour income? Are the different types of capital income taxed in a uniform manner** so as to prevent distortions in the pattern of savings and investment?

■ At the more basic level, a system of well-defined, **secure property rights** and a **sound financial system are important** prerequisites for large-scale private savings. There must be **safe banks and assets in which savings can be placed**. For some poor countries this is a serious issue, and the most essential policy to promote savings in such countries is the establishment of safe property rights and a good financial system. To achieve such a goal may be very difficult if, for instance, there is civil war in the country.

■ Because of its emphasis on capital accumulation, the basic **Solow model strongly suggests that a good system for handling savings and channelling them into productive investment is important for prosperity**.

■ This is an example of a **general and fundamental economic insight**: **institutions are very important for economic performance**, and in some countries an improvement of institutions is the number one need for initiating positive economic development.

The golden rule of saving

■ One may get the **impression** from our discussion that it is good to have **as high a savings and investment rate as possible**. Equation (3.28) certainly says that a higher s gives higher long-run income per worker, y^*. However, it does not require much reflection to see that a **maximal savings rate is not preferable**.

■ **If s is close to one**, GDP per worker will be maximal, but according to (3.29), **consumption per worker will then be close to zero.** If the **ultimate purpose of production is to enjoy the highest possible level of consumption** per worker, what should the savings rate be?

■ Using (3.29) to maximize c^* with respect to s gives the savings rate: $\boldsymbol{s^{**} = \alpha}$. This consumption-maximizing savings rate, s^{**}, is called the **golden rule savings rate**. The capital income share, α, is around 1/3. If policy makers are interested in consumption per person, they should **not try to drive savings rates above 1/3**. Perhaps they should n**ot even go close to that level**, since the **marginal increase in c^* obtained from an increase in s is small if s is already close to α**, and since **higher saving implies postponed consumption** which represents a **welfare cost if people have "time preference"**, preferring to consume now rather than later.

■ Many rich countries in the world have savings and investment rates somewhat above 20 per cent. Some of the East Asian **"growth miracle" countries had investment rates up to around 1/3** at the end of the twentieth century. However, **poor countries typically have much lower savings rates**, down to around 10 per cent, and sometimes even considerably less than that. For the latter countries policies that could work to increase savings rates would seem to be important for raising the long-run level of economic welfare.

Economic growth in the basic Solow model

■ The **macroeconomic dynamics outside steady state** in the Solow model are of interest since this is where **there can be economic growth**.

Transitory growth

■ The **long-run prediction of the Solow model is the steady state**, and in steady state there is **no growth in GDP per worker**. There **can be growth in the GDP itself, but only at the speed of population growth**.

■ **Why is it that growth has to cease in the long run** according to the basic Solow model? Here, **diminishing returns to capital** plays a crucial role. The production function for output per worker, $y_t = Bk_t^{\alpha}$, has a "marginal product", $\alpha Bk_t^{\alpha-1}$, which decreases with k_t because $\alpha < 1$.

■ **Assume that the economy is initially below steady state**, $k_t < k^*$. Production and income per worker will be Bk_t^{α} in period t, and the resulting gross savings per worker, sBk_t^{α}, will exceed the amount $(n + \delta)k_t$ needed to maintain capital per worker in the face of population growth and depreciation (look again at the Solow diagram in the lower part of Figure 3.5). Hence **capital per worker will increase** from one period to the next. As k_t increases, **each unit of additional capital per worker will generate ever smaller increases in income and gross savings per worker** (due to diminishing returns to capital). **At the same time the additional savings needed to compensate for population growth and depreciation increase**, since k_t is increasing.

■ Therefore, ultimately **(in steady state) gross savings per worker**, sBk_t^{α}, will **just cover what is needed to keep capital per worker unchanged**, $(n + \delta)k_t$. In the Solow diagram **diminishing returns is reflected by the function** sBk_t^{α} being curved **with a decreasing slope**.

■ As the steady state is approached, both k_t and y_t will be growing. Hence, **in the basic Solow model there is transitory growth in GDP per worker on the way to steady state**, but no growth in steady state. During the transition phase, growth in k_t goes hand in hand with growth in y_t. As noted earlier, with the assumed aggregate production function, growth in GDP per worker has to be rooted in growth in capital per worker.

An example

■ It is relevant to investigate **how fast or slow transitory growth** is according to the basic Solow model. **If growth fades within a few years, the model could not deliver much of an explanation of growth over decades**.

■ Consider an economy that is initially described by the following parameter values: $B = 1$, $\alpha = 1/3$, $\delta = 0.05$, $n = 0.03$ and $s = 0.08$. The **value of B is just a normalization**, while α and δ have been set at reasonable values, for δ assuming a period length of one year. The values for **n and s**, respectively, **correspond to a high annual population growth rate and a low propensity to save, which could be descriptive of a typical poor country**. We assume that the **economy is initially in steady state** at these parameter values. Using (3.27) and (3.28) one finds steady state capital per worker and GDP per worker, respectively: $k^* = y^* = 1$.

■ Now **suppose** that from an initial year one, the **savings rate increases permanently** up to a level typically seen in rich countries, say, $s' = 0.24$, while the other parameters stay unchanged. The new steady state is easily computed: $k^{*\prime} = 5.20$, and $y^{*\prime} = 1.73$. In the long run, an **increase in the savings rate by 200 per cent implies a 420 per cent increase in capital per worker and a 73 per cent increase in GDP per worker**. But **how fast is the transition** to the new steady state?

■ We can **simulate the transition equation** (3.23) with the savings rate equal to s' and the other parameters as given above over a number of periods $t = 2, 3, ...$, starting with $k_0 = k_1 = 1$. (In year one capital per worker will be at the old steady state level, since the increased savings rate during period one only has an effect on capital per worker from period two.) In this way a sequence, (k_t), can be constructed.

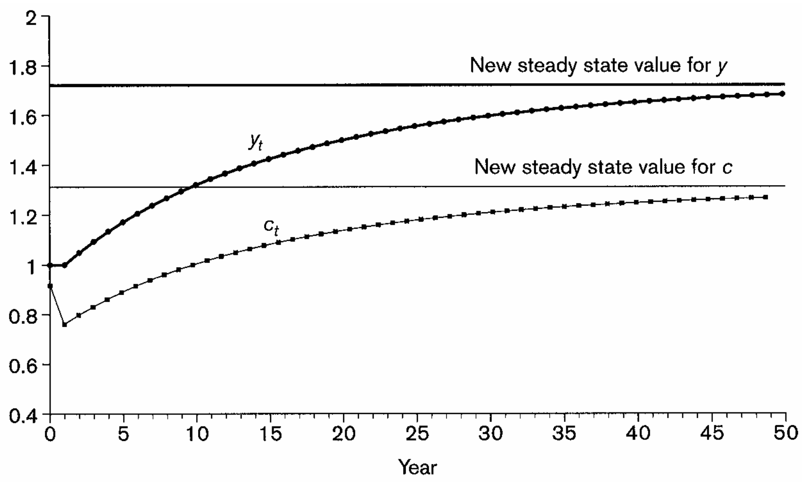
■ From (k_t) one can derive a corresponding sequence (y_t) by using $y_t = k_t^{\alpha}$ in all years, and a sequence (c_0) by using $c_0 = (1 - s)y_0$ and $c_t = (1 - s')y_t$ for $t > 0$. For each year one can then compute the growth rate in GDP per worker, $y_t/y_{t-1} - 1$, or approximate by $g^y = \ln y_t - \ln y_{t-1}$. These operations are most easily done on a **computer using a spreadsheet**.

■ The simulation gives rise to the time series shown in Table 3.1, and illustrated graphically in Figure 3.9, which depicts the evolution in y_t, c_t, and g_t^y over some 50 years after the change in the savings rate.

Table 3.1: Simulation output

Year	k_t	t_t	c_t	g_t^y, exact (%)*	g_t^y, approx (%)**
0	1.000	1.000	0.920		
1	1.000	1.000	0.760	0.00	0.00
2	1.155	1.049	0.797	4.93	4.81
3	1.310	1.094	0.832	4.28	4.19
4	1.463	1.135	0.863	3.76	3.69
5	1.614	1.173	0.892	3.33	3.27
6	1.762	1.208	0.918	2.97	2.92
7	1.907	1.240	0.942	2.66	2.63
8	2.048	1.270	0.965	2.40	2.38
9	2.184	1.298	0.986	2.18	2.16
10	2.317	1.323	1.006	1.98	1.97
⋮	⋮	⋮	⋮	⋮	⋮
∞ (new st.st.)	5.196	1.732	1.316	0	0

Note: *computed as $((y_t - y_{t-1})/y_t) \times 100$, **computed as $(\ln y_t - \ln y_{t-1}) \times 100$


2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
New steady state value for y
y_t
New steady state value for c
c_t
0
5
10
15
20
25
30
35
40
45
50
Year

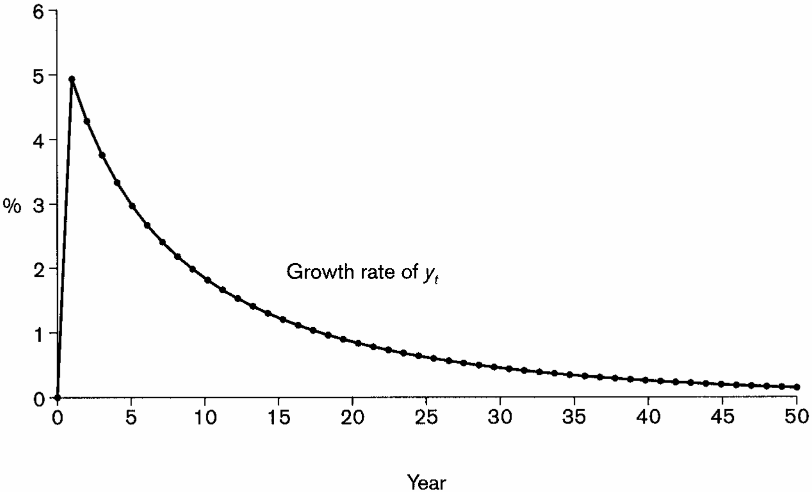


Figure 3.9: The evolution of y_t, c_t and g_t^y after the increase in s

■ There are several interesting features to note from Figure 3.9. It **illustrates the growth perspectives**, according to the Solow model, **for a poor country** bringing its **savings rate up** to the standards of rich countries. For instance, there is a **substantial initial sacrifice of consumption for some years** (six to be precise).

■ Furthermore, there is **considerable transitory growth for a long time**. It takes around **12 years to get half the way** from the old to the new steady state level of GDP per capita. **After 20 years the annual growth rate of GDP per worker is still close to 1 per cent**, and this is still **purely transitory growth** arising from the permanent increase in the savings rate 20 years earlier.

■ **Transitory growth in the Solow model takes place over decades** for realistic parameter values. This **relative slowness in the adjustment** outside steady state is why it is justified to say that: the **basic Solow model *is* a growth model**.

■ From a Solow diagram, such as the lower part of Figure 3.5, it is evident that the relative sizes of the **parameters α, n and δ are decisive for how fast convergence to steady state** will be. The **higher the value of α**, the **less curved and the closer to a straight line the savings curve**, sBk_t^{α}, will be. The **smaller $n + \delta$** is, the **lower will be the position of the ray**, $(n + \delta)k_t$. Intuitively from the figure, **a high and relatively flat sBk_t^{α} curve** and a **low $(n + \delta)k_t$ ray** will imply **a long transition period and hence slow convergence**.

■ A "cousin" to the Solow diagram is useful for illustrating the growth process outside steady state. Dividing by k_t on both sides of (3.26) gives the **modified Solow equation**:

$$\frac{k_{t+1} - k_t}{k_t} = \frac{1}{1+n}[sBk_t^{\alpha-1} - (n - \delta)]$$

■ **In the modified Solow diagram**, (Figure 3.10), the **growth rate of the capital intensity is thus proportional to the vertical distance between the decreasing curve, $sBk_t^{\alpha-1}$, and the horizontal line at $n + \delta$**.

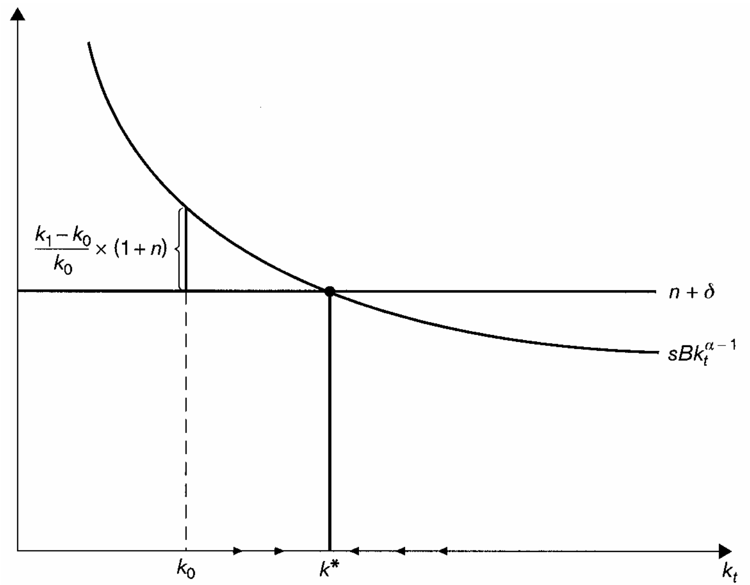


Figure 3.10: The modified Solow diagram

■ The steady state is, of course, where the curve and the line intersect.

■ It follows that the **growth rate in k_t will be higher (in absolute value) the further away from the steady state k_t is**. The same will be true for the growth rate in y_t, since $g_t^y = \alpha g_t^k$. In the numerical example above this feature appeared. **After the initial increase in the savings rate, the growth rate of y_t jumped up and then decreased monotonically back towards zero**.

■ The **model feature that growth is higher the further below steady state the economy is accords with the idea of conditional convergence** discussed earlier. In a modified Solow diagram such as Figure 3.10, the positions of the curve and the line depend on the structural characteristics of the economy. If one country has a higher savings rate than another, the countries being otherwise structurally alike, the decreasing curve of the first country will be situated above that of the second country. If the two countries start out with the same GDP per worker, the first country will grow faster and converge to a higher final level of GDP per worker. However, conditional on countries being structurally alike, the Solow model predicts that countries will converge to the same level of GDP per worker through a growth process where the **countries that are initially most behind will grow the fastest** (consider **the Baltic states** in 1995-2007). In other words, the **Solow model supports conditional convergence**.

Endogenous population growth and club convergence in the basic Solow model

■ Consider a growth model that consists of the same equations as the basic Solow model except that the ***n*** in 3.19 is **no longer exogenous**, but rather an **endogenous variable that depends on prosperity**. Hence, ***n*** **is replaced with** $\boldsymbol{n_t}$, and an additional equation, $n_t = h(y_t) = h(Bk_t^{\alpha})$, or just $n_t = n(k_t)$, states how the population growth rate depends on income per capita and hence on the capital-labour ratio. The Solow equation that can be derived from the model defined this way is:

$$k_{t+1} - k_t = \frac{1}{1+n(k_t)}[sBk_t^{\alpha} - (n(k_t)+\delta)k_t]$$

and that the modified Solow equation is:

$$\frac{k_{t+1}-k_t}{k_t}=\frac{1}{1+n(k_t)}[sBk_t^{\alpha-1}-(n(k_t)+\delta)]$$

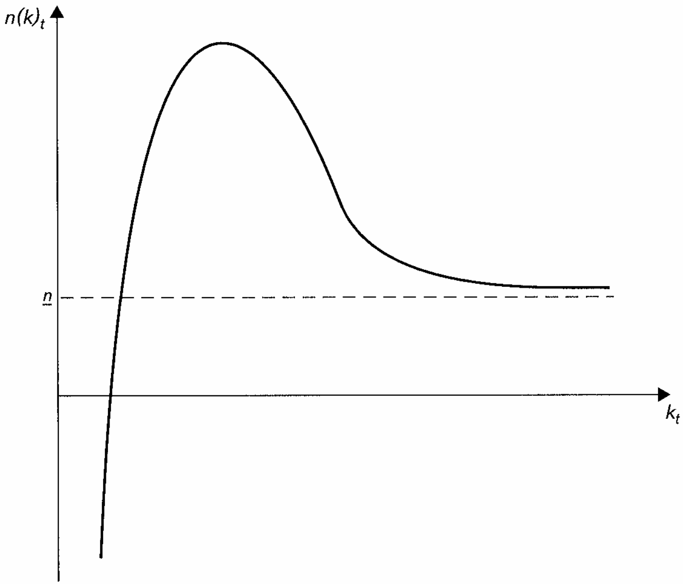


Figure 3.11: Dynamics of *n*

■ It seems reasonable to assume a shape of the function $n(k_t)$ as illustrated in the Figure 3.11. **For low values of k_t, the function $n(k_t)$ is perhaps negative, but increasing**, since higher k_t and y_t mean **better living conditions, less infant mortality**, etc. This is true for k_t **up to a certain value**. From that value, $n(k_t)$ is decreasing, because **as people get richer birth rates fall** for various reasons. However, as k_t increases from the limiting value, $n(k_t)$ never falls below a certain positive value, n in the figure.

■ With such a shape of $n(k_t)$, there may be **three steady state** values for k_t, that is, **three intersections between the curve $sBk_t^{\alpha-1}$ and the curve $n(k_t) + \delta$**. If this is the case, how does the (unmodified) Solow diagram look? **Which of the steady states are stable** in the sense that if k_t departs a little from the steady state in question, the dynamics of the model tend to bring it back towards the steady state, and **which one is unstable?**

■ As shown in Figure 3.12, the investment requirement line crosses the savings curve at points A, B, and C.

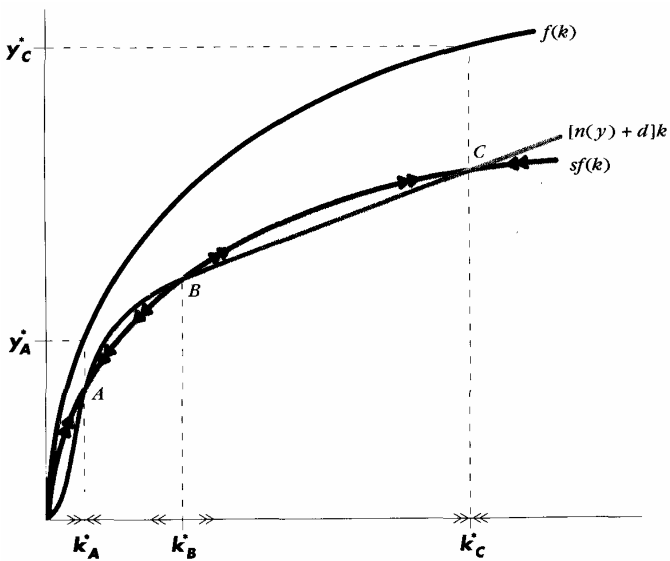


Figure 3.12: The poverty trap

■ Point *A* is a **poverty trap** with high population growth and low income. The equilibrium at *C* has low population growth and high income. Note the arrows showing the direction of movement toward the steady state. Points *A* and *C* are said to be **stable equilibria** because the economy moves toward these points. *B* is an **unstable equilibrium** since the economy tends to move away from *B*.

■ **How can an economy escape from the low-level equilibrium?** There are two possibilities. If a country can put on a **"big push"** which raises income past point *B*, the economy will continue on its own the rest of the way to the high-level point *C*. Alternatively, a nation can effectively **eliminate the low-level trap by moving the savings curve up or the investment requirement line down** so that they **no longer touch at *A* and *B***.

■ **Raising productivity** or raising the **savings** rate **raises the savings line**. **Population control** policies **lower the investment requirement line**.

Summary

■ The basic Solow model studied in this lecture is characterized by the following features:

a. Aggregate output, GDP, is produced from aggregate capital and aggregate labour according to a Cobb-Douglas production function with constant returns to capital and labour and constant total factor productivity.
b. Competitive market clearing ensures that the amounts of capital and labour used in production in each period are equal to the predetermined available amounts and that factors are rewarded at their marginal products. An implication is that the income shares of capital and labour are equal to the elasticities of output with respect to capital and labour, respectively.
c. Capital evolves over time such that from one year to the next, the change in the capital stock equals gross saving (or investment) minus depreciation on the initial capital stock.

d. In each period gross saving is an exogenous fraction of total income or GDP. Depreciation on capital is given by an exogenous depreciation rate. The labour force changes over time at an exogenous population growth rate.

■ The key assumptions of a Cobb-Douglas aggregate production function and a given rate of saving and investment are motivated empirically by the relative constancy of income shares in the long-run and the relative long-run constancy of the GDP share of total consumption (private plus public), respectively.

■ The model implies that capital per worker and output per worker converge to particular constant values in the long run. The convergence point defines the economy's steady state. In steady state, consumption per worker, the real interest rate, and the wage rate are constant as well. The steady state has some empirical plausibility with respect to the predicted value for the long-run real interest rate and with respect to the long-run dependence of GDP per worker on the investment rate and on the population growth rate.

■ The expressions for the steady state values of the key variables contain some sharp predictions of how income per worker and consumption per worker depend on underlying parameters in the long run. These predictions are of importance for the design of structural policies to raise the standard of living. According to the model, policies to make a nation richer should mainly be policies that can increase the investment share of GDP and bring population growth under control, or policies to improve technology. To create a high level of consumption per person, the savings and investment rate should not exceed the golden rule value, which is equal to the elasticity of output with respect to capital.

■ In the model's steady state there is no positive growth in GDP per worker, consumption per worker, or the real wage rate. This is at odds with the stylized facts of growth (in developed economies). Remedying this shortcoming is one of the main purposes of the growth models to be presented later. These models will contain some form of technological progress, such that total factor productivity increases over time. The basic Solow model does give rise to (positive or negative) growth in output per worker during the transition to steady state. This transitory growth is relatively long-lasting for plausible parameter values. Furthermore, the process of transitory growth is such that the further below steady state an economy currently is, the faster it will grow. This is in accordance with the observed conditional convergence between the countries of the world.